

Accelerating beams

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We demonstrate that any two-dimensional accelerating beam can be described in a canonical form in Fourier space. In particular, we demonstrate that there is a one-to-one correspondence between complex functions in the real line (the line spectrum) and accelerating beams. An arbitrary line spectrum can be used to generate novel accelerating beams with diverse transverse shapes. The line spectra for the special cases of the families of Airy and accelerating parabolic beams are provided. © 2009 Optical Society of America
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Accelerating beams have the unusual ability to remain diffraction free while undergoing a quadratic transverse shift during propagation. Until now only two kinds of two-dimensional accelerating beams are known, Airy beams (AiB) [1] and accelerating parabolic beams (ApB) [2]. These beams have been experimentally generated by using liquid-crystal displays (LCDs) [3,4] and by nonlinear crystals [5]. Owing to their special properties, AiBs and ApBs have found important uses in optically mediated particle clearing [6] and in the generation of curve plasma channels in air [7].

In this Letter we introduce the complete theory of the two-dimensional accelerating beams and discuss their mathematical and physical properties. In particular, we demonstrate that there is a one-to-one correspondence between complex functions in the real line (the line spectrum) and accelerating beams. Our results allow the generation of accelerating beam with novel transverse distributions, broadening its possible application even further.

To begin the analysis we must consider the normalized paraxial wave equation

$$\nabla^2 \psi + i \partial_s \psi = 0, \quad (1)$$

where $\nabla^2 = \partial_{uu} + \partial_{vv}$, $(u, v) = (x, y)/\kappa$ represents the dimensionless transverse coordinates, κ is an arbitrary transverse scale, $s = z/2k\kappa^2$ is the normalized propagation distance, and k is the wave number.

It is known that the desired properties of the beam, that is, to remain diffraction free and to exhibit a transverse translation during propagation, can be expressed by

$$|\psi(u, v, s)| = |\psi(u + \delta(s), v, 0)|, \quad (2)$$

where $\delta(s)$ is a unknown function of s that describes the transverse translation of the beam during propagation. The propagation of the beam $\psi(u, v, s)$ is formally given by

$$\psi(u, v, s) = e^{-is\mathbf{p}^2} \psi(u, v, 0), \quad (3)$$

where $\mathbf{p}^2 = -\nabla^2$ and $\exp(-is\mathbf{p}^2)$ is the differential representation of the Huygens diffraction integral.

For the one-dimensional case, in [8], Unnikrishnan *et al.* found that, if and only if $\delta(s) = -s^2$, there is a

unique decomposition of the evolution operator that achieves the desired conditions. Generalizing this work to the two-dimensional case, we find that the unique decomposition of the evolution operator is

$$e^{-is\mathbf{p}^2} = e^{-i2s^3/3} e^{isu} e^{-is^2 p_u} e^{-is\hat{H}}, \quad (4)$$

where $p_u = -i\partial_u$ and

$$\hat{H} = -(\partial_u^2 + \partial_v^2) + u, \quad (5)$$

and we have ignored any possible trivial rescaling and translation. From this unique decomposition we can see that the diffraction-free and accelerating conditions given by Eq. (2) are satisfied if and only if $F(u, v) = \psi(u, v, 0)$ is a solution of

$$\hat{H}F(u, v) = \lambda F(u, v), \quad (6)$$

and $\delta(s) = -s^2$. Therefore there exists a one-to-one mapping between accelerating solutions of the paraxial wave equation and solutions of the two-dimensional linear potential Schrödinger equation. This is important, because if we find all the solutions to the linear potential Schrödinger equation, we can find all the possible accelerating beams.

In order to find solutions to $\hat{H}F = \lambda F$ notice that solutions with eigenvalue λ are given by $F_0(u - \lambda, v)$, where $F_0(u, v)$ is the $\lambda = 0$ solution. Therefore any accelerating beam must have the following structure:

$$\psi(u, v, s) = e^{is(u - \lambda - s^2)} e^{is^3/3} F_0(u - \lambda - s^2, v). \quad (7)$$

To find solutions to the equation $\hat{H}F_0 = 0$, we can try the separation of variables technique. In doing this we will find that this equation is only separable in Cartesian coordinates and parabolic coordinates corresponding to Airy and accelerating parabolic-beam solutions, respectively [2,9]. However, we are interested in intrinsic characteristic of the accelerating beams, and therefore a more general approach must be used.

By using the Fourier transform pair

$$F_0(u, v) = \frac{1}{2\pi} \int e^{ik \cdot r} \tilde{F}_0(k_u, k_v) d\mathbf{k}, \quad (8a)$$

$$\tilde{F}_0(k_u, k_v) = \frac{1}{2\pi} \int e^{-i\mathbf{k} \cdot \rho} F_0(u, v) d\rho, \quad (8b)$$

where $\rho=(u, v)$, $\mathbf{k}=(k_u, k_v)$ are the spacial frequency coordinates, we find that $\hat{H}F_0=0$ is satisfied if the Fourier spectrum \tilde{F}_0 of F_0 satisfies

$$(k_u^2 + k_v^2 + i\partial_{k_u})\tilde{F}_0(k_u, k_v) = 0. \quad (9)$$

It is important to notice that the coordinate k_v is not dynamical in Fourier space. Therefore Eq. (9) is a simple first-order differential equation in k_u whose more general solution is given by

$$\tilde{F}_0(k_u, k_v) = L(k_v) \exp(ik_v^2 k_u + ik_u^3/3), \quad (10)$$

where $L(k_v)$ is an arbitrary complex function on the real line. Finally we can state the following:

Theorem. *A solution of the paraxial wave equation is an accelerating beam if and only if it can be expressed in the following form:*

$$\begin{aligned} \psi(u, v, s) = & e^{is(u-\lambda-s^2)} e^{is^3/3} \\ & \times \mathcal{F}^{-1}[\exp(ik_u k_v^2 + ik_u^3/3) L(k_v)] \\ & (u - \lambda - s^2, v), \end{aligned} \quad (11)$$

where $L(k_v)$ is an arbitrary complex function on the real line.

This theorem is the main result of this Letter. It implies that there is a one-to-one correspondence between complex functions on the real line $L(k_v)$ and accelerating beams. Therefore it is appropriate to call the function $L(k_v)$ the line spectrum of the accelerating beam. To understand this better we can say that the line spectrum is to an accelerating beam as the angular spectrum is to a nondiffracting beam [10].

Now we will examine some important consequences of our analysis. Because the map between solutions of $\hat{H}F=\lambda F$ [Eq. (6)] and the accelerating solutions is given by the unitary evolution operator, the orthogonality properties of the former solutions will be mapped to the latter. Because \hat{H} is a Hermitian operator its eigenvalues λ are real and its eigensolutions orthogonal. Therefore, the accelerating beams with different eigenvalue λ are orthogonal. Notice from Eq. (7) or Eq. (11) that accelerating beams with different λ are just u -axis translations of beams with $\lambda=0$.

It is important to note that, owing to the factor $\exp(-is\lambda)$ in Eq. (11), if we superpose accelerating beams with different λ they will interfere on propa-

gation, destroying their nondiffractive properties. This is similar to the interference of nondiffractive beams with different transverse wave vector k_t . Therefore, if we want to superpose accelerating beams in a coherent way, i.e., without losing its nondiffractive properties, they must have the same λ .

Also, as we mention above, solutions of $\hat{H}F=\lambda F$ are related by a translation and Fourier transformation to the line spectrum $L(k_v)$. Because the translation and Fourier transformation are also unitary operations, the orthogonality properties $L(k_v)$ on the real line will be mapped to orthogonality properties of the accelerating beams on the two-dimensional transverse plane.

In summary, any complete family of orthogonal function on the real line will produce a complete and orthogonal family of accelerating beams. Although this shows that there is an infinite number of orthogonal and complete families of accelerating beams, it is important to emphasize that these beams are defined up to a Fourier transformation. Because Eq. (9) is separable only in Cartesian and parabolic coordinates, the Airy and accelerating parabolic beams are the only complete and orthogonal families of accelerating beams that have explicit closed-form solutions in an orthogonal coordinate system.

Table 1 shows the function $F_0(u, v)$, the line spectrum $L(k_v)$, and the orthogonality properties of the accelerating parabolic, AiBs, and Airy-plane-wave beams (AipwB). The beams are constructed from $F_0(u, v)$ by using Eq. (7), or from $L(k_v)$ by using Eq. (11). $F_0(u, v)$ is related to $L(k_v)$ by Eqs. (10) and (8a). In the orthogonality integral of Table 1, Ψ' means that the beam is evaluated at the prime parameters. $\Theta_n(\cdot)$ denotes the square integrable eigensolutions of the quartic oscillator equation, i.e., the one-dimensional Schrödinger equation with potential $V(x)=x^4/4$ [2] and (η, ξ) are parabolic coordinates. In the case of the AiBs, β corresponds to translations of the beam in the v axis. The complete family of AiBs and ApwB are introduced for the first time to our knowledge in this work.

Figure 1 shows several transverse-field distributions of accelerating beams with their corresponding line spectra $L(k_v)$. As we can see in Figs. 1(d) and 1(e) novel accelerating beams can be created by selecting different line spectra $L(k_v)$.

It is easy to see that any accelerating beam carries infinity energy, because its Fourier spectrum Eq. (10) is not bounded in the k_u coordinate. This implies that the transverse-field distribution of any accelerating beam is not strongly confined, i.e., it does not decay

Table 1. Orthogonal Families of AiB, ApB, and AipwB and Their Line Spectra

Beam	$F_0(u, v)$	$L(k_v)$	$\iint \Psi^* \Psi' du dv \propto$	Comment
AiB	$Ai[(u-v+\beta)/2^{2/3}]$ $\times Ai[(u+v-\beta)/2^{2/3}]$	$e^{-i\beta k_v}$	$\delta(\beta-\beta')\delta(\lambda-\lambda')$	$\beta \in \mathbb{R}, \lambda \in \mathbb{R}$
ApB	$\Theta_n(\eta)\Theta_n(i\xi)$	$\Theta_n(\sqrt{2}k_v)$	$\delta_{nn'}\delta(\lambda-\lambda')$	$n=0, 1, 2, \dots, \lambda \in \mathbb{R}$ $(u, v)=(\eta^2/2-\xi^2/2, \eta\xi)$
AipwB	$Ai(u+\omega^2)\exp(i\omega v)$	$\delta(k_v-\omega)$	$\delta(\omega-\omega')\delta(\lambda-\lambda')$	$\omega \in \mathbb{R}, \lambda \in \mathbb{R}$

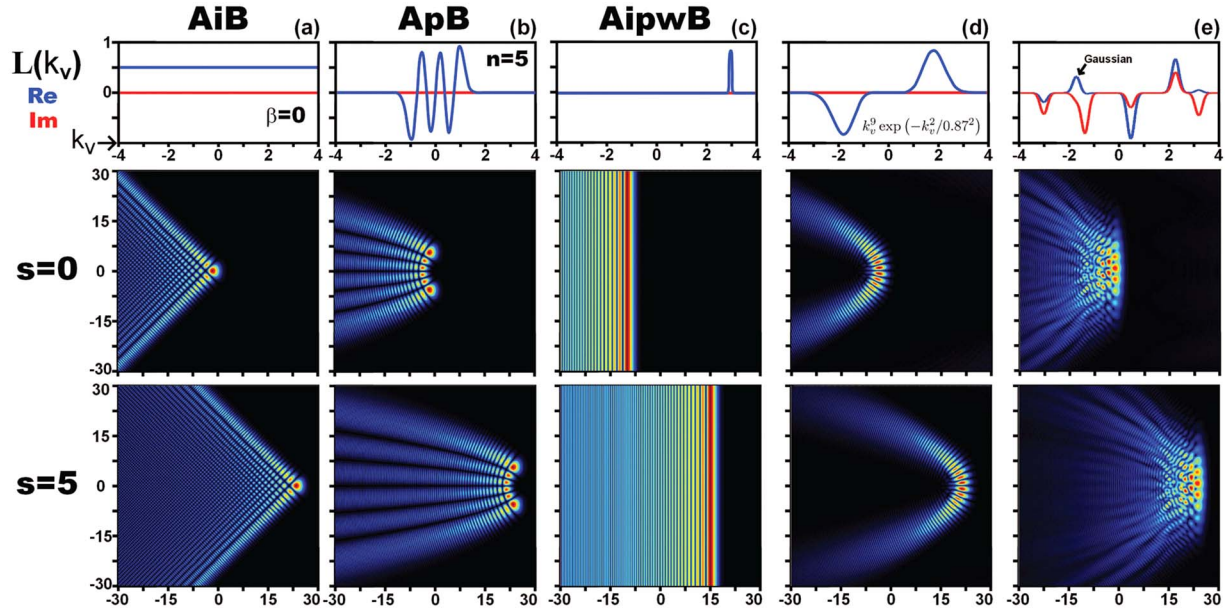


Fig. 1. (Color online) Transverse amplitude distributions of some accelerating beams at $s=0,5$ and their corresponding $L(k_v)$ line spectra.

faster than $1/x$ for some direction in the transverse plane. Their infinite oscillating tail that extends in the negative u axis can be appreciated in Fig. 1. The ApwBs correspond to the tensor product of a one-dimensional AiB and a plane wave. But note that the ApwBs are confined only in the $+u$ direction, while the AiBs and accelerating beams are unconfined only in the $-u$ direction.

We will now construct a finite-energy version of the accelerating beams that can be physically realizable and that exhibit the unusual properties of the ideal accelerating beams over a finite propagation distance. The construction of finite-energy solutions is straightforward if we use the results of Bandres and Guizar-Sicairos in [11], where they showed how to obtain the closed-form propagation of any paraxial beam through an ABCD optical system if the free-space propagation is known. In particular, we are interested in the propagation of accelerating beams that have an exponential apodization in the u axis; this will eliminate the infinite oscillating tail of the accelerating beams and will make them finite. Therefore finite-energy accelerating beams are given by

$$\Psi(u, v, s) = e^{ua} \psi(u + 2isa, v, s),$$

$$\propto \psi(u + a^2, v, s - ia),$$

where ψ is any accelerating beam given by Eq. (11). The Fourier spectrum of the finite-energy accelerat-

ing beams at the plane $z=0$ is given by

$$\mathcal{F}[\Psi](k_u, k_v) = L(k_v) e^{ik_v^2(k_u + ia) + i(k_u + ia)^3/3}.$$

Accelerating beams for an arbitrary line spectrum $L(k_v)$ can be easily generated by encoding this spectrum onto an LCD, as was done for the ApB in [4].

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